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On superplanckian scattering on the brane

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Abstract. The multidimensional space-time with $(D-4)$ compact extra space dimensions and SM fields confined on a four-dimensional brane is considered. The elastic scattering amplitude of two particles interacting by gravitational forces is calculated at superplanckian energies. Particular attention is paid to a proper account of zero (massless) graviton mode. The renormalized Born pole is reproduced in the eikonal amplitude which makes a leading contribution at small momentum transfers. This singular part of the amplitude coincides with well-known D-dimensional amplitude taken at $D \to 4$. The expression for the contribution from massive graviton modes to the eikonal is derived, and its asymptotics in the impact parameter is calculated. Our formula gives the correct four-dimensional expression at $R_c \to 0$, where R_c is the radius of the higher dimensions. The results are also compared with those obtained previously for the scattering of the bulk fields in flat extra dimensions.

1 Introduction

In the four-dimensional space-time gravity is very weak as compared with the interactions of the standard model (SM) fields. Namely, the Newton constant is equal to $G_N = M_{Pl}^{-2}$, where $M_{Pl} = 1.2 \cdot 10^{19}$ GeV is the Planck mass, while the electroweak scale is about $m_{EW} \sim 10^3$ GeV. In order to explain the huge ratio of the two physical scales in nature, M_{Pl}/m_{EW}, a scheme with additional space dimensions with a flat metric has been proposed [1] (in what follows, referred to as the ADD model). All d extra dimensions are compact with the radius R_c. In other words, the space-time is $R^4 \times M_d$, where M_d is a d-dimensional manifold volume R_c^d. If $R_c^{-1} \ll m_{EW}$, a gravitational potential will get negligible corrections at distances $r \gg R_c$.

Let M_D be the fundamental Planck scale in D-dimensional theory $(D = 4 + d)$. Then it can be shown [1] that

$$M_{Pl}^2 = R_c^d M_D^{2+d} \,, \tag{1}$$

or, equivalently,

$$R_c = 2 \cdot 10^{31/d-17} \left(\frac{1\,\text{TeV}}{M_D} \right)^{1+2/d} \text{cm} \,. \tag{2}$$

One can get $M_D \sim 1$ TeV, if the compactification radius R_c is large enough. The radius R_c depends on d and it ranges from 1 mm to 1 fm if d runs from 2 to 6. Since $R_c \gg m_{EW}^{-1}$, all standard model (SM) gauge and matter fields are to be confined to a three-dimensional brane embedded into the $(3+d)$-dimensional space (gravity alone lives in the bulk).

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From the point of view of the four-dimensional space-time, there arises a Kaluza–Klein (KK) tower of massive graviton modes, $G_{\mu\nu}^{(n)}$, with masses

$$m_n = \frac{\sqrt{n^2}}{R_c} \,, \quad n^2 = n_1^2 + n_2^2 + \ldots + n_d^2 \,, \tag{3}$$

where n defines the KK excitation level. So, a mass splitting is $\Delta m \sim R_c^{-1}$ and we have an almost continuous spectrum of gravitons.

The interaction of the gravitons with the SM fields is described by the Lagrangian [1]

$$\mathcal{L} = -\frac{1}{\bar{M}_{Pl}} G_{\mu\nu}^{(n)} T^{\mu\nu} \,, \tag{4}$$

where $\mu, \nu = 0, 1, 2, 3$ and $\bar{M}_{Pl} = M_{Pl}/\sqrt{8\pi}$ is the reduced Planck mass. One can conclude from (4) that the coupling of both massless and massive graviton is universal and very small $(\sim 1/\bar{M}_{Pl})$. However, the multiplicity of the KK states produced in high-energy collisions is huge and it is equal to $(\sqrt{s}R_c)^d$, where $\sqrt{s}$ is the collision energy. The typical cross section for a process involving the production of the KK graviton excitations with masses $m_n \le \sqrt{s}$ is suppressed only by the scale M_D:

$$\sigma_{KK} \sim \frac{s^{d/2}}{M_D^{d+2}} \,. \tag{5}$$

So, the ADD model can be tested at future hadronic colliders and at e^+e^- linear colliders in the range of TeV energies (the Planck regime). There are a lot of papers on collider phenomenology within the framework of the extra dimensions. The interested reader can find references in the reviews in [2].

Collisions in the transplanckian regime ($\sqrt{s} \gg M_D$) were considered in a variety of papers in the framework of string theory [3], in the eikonal approximation of the reggeized graviton exchange [4], as well as in different four-dimensional approaches [5, 6]. The equivalence of various schemes has been demonstrated in [7]. Note that in [3, 4] a collision of the *bulk fields* in D-dimensional *flat space-time* was considered.

In [8] an estimate of the high-energy gravitational cross sections of hadrons has been made. The contribution from the KK excitations of the graviton changes the t-channel propagator $(-t)^{-1}$ by

$$\frac{1}{-t} \to \sum_{n_1^2+\dots n_d^2 \geqq 0} \frac{1}{-t + \sum\limits_{i=1}^{d} \dfrac{n_i^2}{R_c^2}} \,. \tag{6}$$

It was argued in [8], that the range $n \lesssim n_{\max} \sim M_s R_c$, where M_s is a quantum gravity (string) scale, makes a dominant contribution to the hadronic cross sections. Using the replacement ($d > 2$)

$$\sum_{n_1^2+\dots n_d^2 \geqq 0} \frac{1}{-t + \sum\limits_{i=1}^{d} \dfrac{n_i^2}{R_c^2}} \to R_c^2 \int \mathrm{d}^d \Omega \int\limits_0^{n_{\max}} \mathrm{d}n \, n^{d-3} \,, \tag{7}$$

the following results for the total cross section have been obtained (after unitarization):

$$\sigma_{\mathrm{tot}}(s) \simeq \frac{4\pi s}{M_D^4} \,. \tag{8}$$

However, the papers of [8] have unjustified approximations, as it was pointed out in [9]. In particular, the presence of the massless exchange quantum (zero mode of the graviton) should result in *infinite elastic and total hadronic cross sections*, contrary to the equality (8). For strong interactions without gravitational forces, the upper (Froissart) bound for $\sigma_{\mathrm{tot}}(s)$ is modified by an additive term $(\pi r_c / m_\pi) \ln s$, if the extra dimensions are compactified onto a circle with the radius r_c [10].

Recently, results on a collision of *brane particles*, which interact by graviton forces in the ADD model with the compact extra dimensions, have been presented in [11, 12]. These papers consider the approximation that the size of the extra dimensions, R_c, is effectively infinite. In [13] the results from [12] were applied for calculations of di-jet differential cross sections at the LHC energy in a kinematical region where gravity dominates.

In the present paper we will be interested in the effects of finite R_c. We have to go beyond the approximation used in [8, 12]. In particular, the massless graviton mode should be properly taken into account. That is why, in the present paper, we calculate the scattering amplitude for two particles confined on the brane, by separating the massless graviton contribution from massive graviton effects from the very beginning.

In the next section we review briefly the results on the scattering in both D flat dimensions [3, 4] and four flat dimensions [5, 6]. In the beginning of Sect. 3 we consider the approach proposed in [12]. The rest of Sect. 3 is devoted to calculations of the eikonal amplitude. In the last section we discuss our results and compare them with the results obtained by other authors mentioned in this paper. In the appendix technical details of our calculations are presented.

2 Transplanckian collision in the bulk

In this section we recall some results on transplanckian collisions in models with extra dimensions. As was mentioned in the Introduction, the transplanckian regime has been analyzed in detail in the string theory. String theory has the fundamental classical constant α', its inverse being the string tension. Since the leading graviton trajectory is at $\alpha(t) = 2 + (\alpha'/2)t$, one expects that at high s graviton exchange will dominate the light-string scattering amplitude for any number of loops.

The transplanckian regime is characterized by a strong effective coupling $\alpha_G(s) = G_D s$ ($G_D = M_D^{-(2+d)}$ is the D-dimensional Newton constant). In [3] the four-string scattering amplitude was calculated in the kinematical region

$$\begin{aligned} &\alpha' s \gtrsim (M_D \sqrt{\alpha})^{d+2} \gg 1 \,, \\ &|t| \lesssim \alpha'^{-1} \,, \quad \alpha' |t| \lesssim (\alpha' R_c)^{-2} \,. \end{aligned} \tag{9}$$

The inequalities (9) mean that the tree amplitude is large ($G_D s \alpha'^{-d/2} \gg 1$), while the loop expansion parameter, $G_D \alpha'^{-(1+d/2)}$, is small. Due to the second restriction on t in (9), *compactified momenta are not noticeably excited*. In terms of the impact parameter b, the limitations look like

$$b > \lambda_s \,, \quad b > R_G(s) \,, \tag{10}$$

where $\lambda_s = \sqrt{2\alpha' \hbar}$ is a fundamental quantum length in the string theory and $R_G(s) \simeq (2G_D\sqrt{s})^{1/(d+1)}$ is a gravitational radius.

The leading contribution to the scattering amplitude at the impact parameter b has all powers of $\alpha_G(s)$ and it is the same in all approaches at $b \gg \lambda_{\mathrm{Pl}}, R_G(s)$, where $\lambda_{\mathrm{Pl}} = (\hbar G_D)^{1/(d+2)}$ is the Planck length.

The amplitude is of a classical (eikonal) form. For large b the eikonal function is given by [3]

$$\begin{aligned} \chi(b,s) &\equiv \chi_{\mathrm{ACV}}(b,s) \\ &\simeq \left(\frac{\tilde{b}_c}{b}\right)^d + \mathrm{i}\pi^2 \frac{G_D s \alpha'^{-d/2}}{(\pi \ln s)^{d/2+1}} \exp\left(-\frac{b^2}{4\alpha' \ln s}\right) \,, \end{aligned} \tag{11}$$

where $\tilde{b}_c = \left[\alpha_G(s) 2\pi^{-d/2} \Gamma(d/2)\right]^{1/d}$. As one can see, $\chi_{\mathrm{ACV}}(b,s)$ *has both a real and an imaginary part*. The former has a power-like behavior in b, while the latter decreases exponentially at $b \gg 2\alpha' \ln s$. Correspondingly, at small t (namely, at $|t| \lesssim \alpha_G(s)^{-2/d}$) the amplitude has the asymptotics

$$A^{\rm eik}_{\rm ACV}(s,t) \simeq A_{\rm B}(s,t) \qquad (12)$$
$$+\, \mathrm{i}\,\mathrm{const}\frac{(16\pi\alpha_G(s))^2 s}{d(d-2)}\left[-|t|^{d/2-1} + (16\pi G_D s)^{2/d-1}\right].$$

Here

$$A_{\rm B}(s,t) = \frac{8\pi\alpha_G(s)s}{-t} \qquad (13)$$

is the Born amplitude. Thus, *the Born term dominates at small* t.

For large t ($|t| \gg \tilde{b}_{\rm c}^{-2}$) the amplitude has the following behavior [3]:

$$A^{\rm eik}_{\rm ACV}(s,t) \qquad (14)$$
$$\sim \frac{8\pi\alpha_G(s)s}{-t}\exp(\mathrm{i}\phi_D)\left(4\pi\left(\tilde{b}_{\rm c}\sqrt{|t|}\right)^d\right)^{-d/2(d+1)}.$$

The D-dimensional phase

$$\phi_D = \frac{d+1}{d}\left(G_D s\, 2\pi^{-d/2}\Gamma(1+d/2)|t|^{d/2}\right)^{1/(d+1)} \qquad (15)$$

has a pole at $D = 4$. So, the limit $D \to 4$ is completely non-perturbative due to the divergent (Coulomb) phase.

In [4] the same result (14) was obtained by summing multiple reggeized graviton exchange in the eikonal approximation. Although Regge behavior is present at each order, it is absent in the final result (14). It is important to note that the magnitude of the *scattering amplitude is defined by a single non-reggeized graviton exchange* (in full analogy with the case of Coulomb scattering dominated by a single photon exchange).

The scattering amplitude may be directly calculated in four dimensions [5,6] (see also [14]):

$$A^{\rm eik}_{\rm HVV}(s,t) = A_{\rm B}(s,t)\frac{\Gamma(1-\mathrm{i}\alpha_G(s))}{\Gamma(1+\mathrm{i}\alpha_G(s))}\left(\frac{4\mu_{\rm IR}^2}{-t}\right)^{-\mathrm{i}\alpha_G(s)}, \qquad (16)$$

or it can be obtained from the D-dimensional expression by taking the limit $D \to 4$ [3,4]. The quantity $\mu_{\rm IR}$ in (16) is an infrared cutoff. It arises in the limit $D \to 4$, when the pole $(D-4)^{-1}$ is interpreted as the logarithm of $\mu_{\rm IR}$ [4]. If the amplitude is calculated as a sum of soft gravitons with a small mass $m_{\rm grav}$, this cutoff is related to the graviton mass, $\mu_{\rm IR} = (1/2)m_{\rm grav}\mathrm{e}^{\gamma}$, where γ is the Euler constant.

Let us note that the eikonal amplitude in quantum electrodynamics can be obtained from (16) by the simple replacement $-\alpha_G(s) \to \alpha_{\rm em} = e_1 e_2/4\pi$, where $e_{1,2}$ are the electric charges of colliding massless charged particles [15]. In such a case, $\mu_{\rm IR}$ is proportional to a regulating photon "mass" [15].

At large z, $|\arg z| < \pi$, the Γ-function has an asymptotics $\Gamma(z) = \sqrt{2\pi}\mathrm{e}^{-z}\mathrm{e}^{(z-1/2)\ln z}\left[1 + \mathrm{O}(z^{-1})\right]$ [16]. Then we obtain from (16) that in four dimensions (see also [4])

$$A^{\rm eik}(s,t)\Big|_{\alpha_G(s)\gg 1} \simeq -A_{\rm B}(s,t)\left(\frac{4\mu_{\rm IR}^2}{-t}\right)^{-\mathrm{i}\alpha_G(s)}$$
$$\times\, \mathrm{i}\exp\left[-\mathrm{i}2\alpha_G(s)(\ln\alpha_G(s) - 1)\right]. \qquad (17)$$

In the next section we will consider the case when colliding particles are confined to the brane, with the gravity living in the bulk. Another difference will be that the compactified momenta become essential, contrary to the approach considered in this section.

3 Transplanckian collision on the brane

In the ADD model, all SM fields live on the (1 + 3)-dimensional brane embedded in the D-dimensional space-time. Thus, their collisions are also confined to the brane. In particular, the impact parameter space is two-fold. On the other hand, in the transplanckian region, where the collision energy $\sqrt{s}$ is much larger than the fundamental gravity scale M_D, but the momentum transfer t is small, the scattering of four-dimensional particles is dominated by the exchange of D-dimensional gravitons.

The (elastic) scattering of two (different) massless particles *living on the brane* in the kinematical region

$$\sqrt{s} \gg M_D\,, \qquad s \gg -t \qquad (18)$$

was first considered in [11] and analyzed in more detail in [12, 13]. The ladder diagrams contributing to a *non-reggeized graviton exchange* in the t-channel were summed in the eikonal approximation [12]. From the point of view of a four-dimensional observer, the massless bulk graviton is represented by a tower of massive gravitons (3). Since the higher space dimensions are compactified with the radius $R_{\rm c}$, one has a sum in the (quantized) momentum transfer in the extra dimensions $q_\perp^{(n)} = n/R_{\rm c}$ instead of an integral in $\mathrm{d}^{D-4}q_\perp$. The Born amplitude is, therefore, of the form

$$A^{\rm B}(s,t) = G_{\rm N}s^2\sum_{n_1^2+\ldots n_d^2\geqq 0}\frac{1}{-t+\sum\limits_{i=1}^{d}\frac{n_i^2}{R_{\rm c}^2}}\,. \qquad (19)$$

Here and in what follows the reduced gravitational constant, $\bar{G}_{\rm N} = \bar{M}_{\rm Pl}^{-2}$ is always assumed; see (4). For simplicity, we will write $G_{\rm N}$ instead of $\bar{G}_{\rm N}$ (and, correspondingly, G_D instead of $\bar{G}_D$). Thus, to compare our results with those of [3–6], one will have to use the substitution $G_{\rm N} \to 8\pi G_{\rm N}$.

In [12] the following replacement was made by assuming that $R_{\rm c}$ is large (compare with (7)):

$$\sum_{n_1^2+\ldots n_d^2\geqq 0}\frac{1}{-t+\sum\limits_{i=1}^{d}\frac{n_i^2}{R_{\rm c}^2}} \to \int \mathrm{d}^d\Omega\int_0^\infty \mathrm{d}l\,l^{d-1}\frac{1}{-t+\frac{l^2}{R_{\rm c}^2}}\,. \qquad (20)$$

As a result, it was obtained that

$$A^{\rm B}_{\rm GRW}(s,t) = \pi^{d/2}\Gamma(1-d/2)\left(\frac{s}{M_D^2}\right)^2\left(\frac{-t}{M_D^2}\right)^{d/2-1}. \qquad (21)$$

At one- and higher-loop levels it is a ladder diagram that makes the leading contribution to the amplitude. The sum of all such diagrams results in the eikonal representation for the amplitude [12]:

$$A^{\mathrm{eik}}_{\mathrm{GRW}}(s,t) = -2\mathrm{i}s \int \mathrm{d}^2 b_\perp \mathrm{e}^{\mathrm{i}q_\perp b_\perp} \left(\mathrm{e}^{\mathrm{i}\chi_{\mathrm{GRW}}(b_\perp)} - 1 \right) , \tag{22}$$

with the eikonal given by

$$\chi_{\mathrm{GRW}}(b_\perp) = \frac{1}{2s} \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} \mathrm{e}^{-\mathrm{i}q_\perp b_\perp} A^{\mathrm{B}}(s, q_\perp^2) . \tag{23}$$

After a substitution of the Born amplitude (21) in (23), one gets [12]

$$\chi_{\mathrm{GRW}}(b) = \left(\frac{b_{\mathrm{c}}}{b} \right)^d , \tag{24}$$

where

$$b_{\mathrm{c}} = \left[\frac{s\,(4\pi)^{d/2-1}\Gamma(d/2)}{2M_D^{d+2}} \right]^{1/d} \equiv 2\sqrt{\pi} R_{\mathrm{c}} \left[\frac{G_{\mathrm{N}} s\, \Gamma(d/2)}{8\pi} \right]^{1/d} . \tag{25}$$

At $d \to 0$, the eikonal $\chi_{\mathrm{GRW}}(b)$, see (24), has the expansion

$$\chi_{\mathrm{GRW}}(b)\Big|_{d\to 0} \simeq \frac{G_{\mathrm{N}} s}{8\pi} \left[\frac{2}{d} + \ln\left(\frac{2R_{\mathrm{c}}}{b} \right)^2 + \Psi(1) + \ln\pi \right] , \tag{26}$$

where $\Psi(z)$ is the Ψ-function [16].

At all intermediate steps (20)–(23), the number of extra dimensions d was regarded as a (non-integer) parameter. The final expression (24) has no divergences in d at $d > 0$, although the Born amplitude has simple poles at $d = 2, 4, \ldots$ (because of the Γ-function in (21)).

In [13] the conclusion was made that "*even for $q = 0$, the scattering amplitude is dominated by $b \sim b_{\mathrm{c}}$ and not by $b = \infty$, as opposed to the Coulomb case. This result follows from the different dimensionalities of the space on which the scattered particles and exchange graviton live*".

However, we will show that the Born amplitude survives after summation of the KK excitations of the graviton and does contribute to the eikonal. In its turn, this means that long-range forces (Coulomb singularity) still are present in the scattering of brane particles.

Indeed, for $d = 1$ the series (19) converges, and it has a pole t^{-1} corresponding to the zero massless mode of the graviton. It would be strange to expect that long-range forces are present for $d = 1$, but disappear when gravity lives in more than one extra dimension. For $d \geqq 2$, the sum (19) is divergent, and it needs regularization. Following [12], we will use the dimensional regularization, by considering d to be non-integer at the intermediate steps of our calculations. The final result will be well defined for all $d \geqq 0$.

Although the change "*summation in n*" → "*integration in* dn" (where n labels the KK excitation level of the graviton) is justified at $R_{\mathrm{c}}\sqrt{|t|} \gg 1$, it should be done more accurately than it was dealt with in (7) and (20). The crucial point is that *a contribution from the zero (massless) graviton mode must be isolated before the replacement* (20):

$$A^{\mathrm{B}}(s,t) = \frac{G_{\mathrm{N}} s^2}{-t} + G_{\mathrm{N}} s^2 \sum_{n_1^2+\ldots+n_d^2 \geqq 1} \frac{1}{-t + m_n^2} . \tag{27}$$

In the case of large extra dimensions, when the mass splitting is small ($\Delta m = 1/R_{\mathrm{c}}$), we can write

$$\sum_{n_1^2+\ldots+n_d^2 \geqq 1} \frac{1}{-t+m_n^2} \to R_{\mathrm{c}}^d \int \mathrm{d}^d\Omega \int_{R_{\mathrm{c}}^{-1}}^{\infty} \mathrm{d}m\, m^{d-1} \frac{1}{-t+m^2} . \tag{28}$$

Then (27) can be recast as follows:

$$\begin{aligned} A^{\mathrm{B}}(s,t) &= \frac{G_{\mathrm{N}} s^2}{-t} \\ &\times \left[1 + \left(\sqrt{|t|} R_{\mathrm{c}} \right)^d \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\left(\sqrt{|t|}R_{\mathrm{c}}\right)^{-1}}^{\infty} \mathrm{d}y y^{d-1} \frac{1}{1+y^2} \right] \\ &\equiv A_0^{\mathrm{B}}(s,t) + A^{\mathrm{B}}_{\mathrm{mass}}(s,t) . \end{aligned} \tag{29}$$

Strictly speaking, the inequality $\sqrt{|t|}R_{\mathrm{c}} \gg 1$ must be satisfied in (28). The physically interesting region is $|t| \gtrsim 0.01\,\mathrm{GeV}^2$. From (2) one can show that R_{c}^{-2} is equal to $0.2 \cdot 10^{-24}\,\mathrm{GeV}^2$, $0.6 \cdot 10^{-9}\,\mathrm{GeV}^2$, and $0.9 \cdot 10^{-6}\,\mathrm{GeV}^2$ for $d = 2$, $d = 4$, and $d = 6$, respectively. Thus, $A^{\mathrm{B}}_{\mathrm{mass}}(s,t)$ approximates well the Born amplitude at $|t| \gtrsim 0.01\,\mathrm{GeV}^2$ and $d \leqq 6$, while $A_0^{\mathrm{B}}(s,t)$ gives the correct singularity at $t \approx 0$.

The integral in the RHS of (29), representing *the contribution from the massive gravitons*, can be calculated and rewritten in the form

$$\begin{aligned} &A^{\mathrm{B}}_{\mathrm{mass}}(s,t) \\ &= \frac{G_{\mathrm{N}} s^2}{-t} \left(\sqrt{|t|} R_{\mathrm{c}} \right)^d \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\left(\sqrt{|t|}R_{\mathrm{c}}\right)^{-1}}^{\infty} \mathrm{d}y y^{d-1} \frac{1}{1+y^2} \\ &= G_{\mathrm{N}} s^2 R_{\mathrm{c}}^2 \frac{\pi^{d/2}}{\Gamma(d/2)(1-d/2)} {}_2F_1\left(1, 1-\frac{d}{2}; 2-\frac{d}{2}; tR_{\mathrm{c}}^2 \right) , \end{aligned} \tag{30}$$

where ${}_2F_1(\alpha_1, \alpha_2; \beta_1; z)$ is the hypergeometric function [16], and we have taken into account the relation

$$R_{\mathrm{c}} = \frac{1}{M_D} \left(\frac{M_{\mathrm{Pl}}}{M_D} \right)^{2/d} = \left(\frac{G_D}{G_{\mathrm{N}}} \right)^{1/d} . \tag{31}$$

Thus, $A^{\mathrm{B}}_{\mathrm{mass}}(s,t)$ converges at $t \to 0$.

We see from (30) that $A^{\mathrm{B}}_{\mathrm{mass}}(s,t) = 0$ for $d = 0$. In such a case, it is the four-dimensional massless graviton that contributes to $A^{\mathrm{B}}(s,t)$, see (29), as it should.

In order to get the asymptotics of the Born amplitude at large t, we use the equivalent expression for $A^{\mathrm{B}}_{\mathrm{mass}}(s,t)$:

$$\begin{aligned} &A^{\mathrm{B}}_{\mathrm{mass}}(s,t) \\ &= \frac{G_{\mathrm{N}} s^2}{-t} \pi^{d/2} \Big[\Gamma(1-d/2) \left(-tR_{\mathrm{c}}^2 \right)^{d/2} \\ &\quad - \frac{1}{\Gamma(1+d/2)} {}_2F_1\left(1, \frac{d}{2}; 1+\frac{d}{2}; \frac{1}{tR_{\mathrm{c}}^2} \right) \Big] . \end{aligned} \tag{32}$$

As it follows from (32), the large t behavior of $A^{\mathrm{B}}_{\mathrm{mass}}(s,t)$ is similar to that of $A^{\mathrm{B}}_{\mathrm{GRW}}(s,t)$, see (21):

$$A^{\mathrm{B}}_{\mathrm{mass}}(s,t)\Big|_{R_c^2|t|\gg 1} \simeq \pi^{d/2}\Gamma(1-d/2)\left(\frac{s}{M_D^2}\right)^2\left(\frac{-t}{M_D^2}\right)^{d/2-1} \times\left[1-\frac{1}{\Gamma(1+d/2)\Gamma(1-d/2)}\,(-tR_c^2)^{-d/2}\right]. \quad (33)$$

It is convenient to divide the "massive" part of the eikonal,

$$\chi_{\mathrm{mass}}(b)=\frac{1}{2s}\int\frac{\mathrm{d}^2q_\perp}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}q_\perp b_\perp}\,A^{\mathrm{B}}_{\mathrm{mass}}(s,t) = \frac{1}{4\pi s}\int_0^\infty q_\perp\mathrm{d}q_\perp\,J_0(bq_\perp)A^{\mathrm{B}}_{\mathrm{mass}}(s,-q_\perp^2)\,, \quad (34)$$

into two parts:

$$\chi_{\mathrm{mass}}\equiv\chi^{(1)}_{\mathrm{mass}}(b)+\chi^{(2)}_{\mathrm{mass}}(b)\,. \quad (35)$$

Here $\chi^{(1)}_{\mathrm{mass}}(b)\equiv\chi_{\mathrm{GRW}}(b)$, see (24), and

$$\chi^{(2)}_{\mathrm{mass}}(b)=-G_{\mathrm{N}}s\,\frac{\pi^{d/2-1}}{8\Gamma(1+d/2)}\,I(b)\,, \quad (36)$$

where $I(b)$ is given by the integral

$$I(b)=\int_0^\infty\frac{\mathrm{d}x}{x}J_0\left(\frac{b}{R_c\sqrt{x}}\right)\,{}_2F_1\left(1,\frac{d}{2};1+\frac{d}{2};-x\right). \quad (37)$$

The integral in (37) cannot be directly expressed in terms of algebraic or special functions. But we will be able to calculate its behavior in impact parameter at both large and small b, if we define $I(b)$ as the limit

$$I(b)=\lim_{\epsilon\to 0}I_\epsilon(b), \quad (38)$$

where we have introduced

$$I_\epsilon(b)=\int_0^\infty\mathrm{d}x x^{-1+\epsilon}J_0\left(\frac{b}{R_c\sqrt{x}}\right)\,{}_2F_1\left(1,\frac{d}{2};1+\frac{d}{2};-x\right). \quad (39)$$

The integral in (39) is well defined at $-3/4<\mathrm{Re}\,\epsilon<1,\mathrm{Re}\,d/2$ (we assume that $\mathrm{Re}\,d>0$). Thus, the limit $\lim_{\epsilon\to 0}I_\epsilon$ exists. Moreover, I_ϵ is a table integral (see formula 2.21.4.6 from [17]):

$$I_\epsilon(b)=\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)} \times\left[-\left(\frac{b^2}{4R_c^2}\right)^\epsilon{}_2F_3\left(\frac{d}{2},1;1+\frac{d}{2},1+\epsilon,1+\epsilon;\frac{b^2}{4R_c^2}\right) + \frac{d}{d-2\epsilon}\Gamma^2(1+\epsilon){}_1F_2\left(\frac{d}{2}-\epsilon;1+\frac{d}{2}-\epsilon,1;\frac{b^2}{4R_c^2}\right)\right], \quad (40)$$

where ${}_pF_q\left(\alpha_1,\dots\alpha_p;\,\beta_1,\dots\beta_q;\,z\right)$ is the generalized hypergeometric function [16].

At $b\ll R_c$, we immediately get from (38) and (40) ($d>0$)

$$I(b)\Big|_{b\ll R_c} \simeq\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{-\left(\frac{b^2}{4R_c^2}\right)^\epsilon\left[1+\frac{1}{1+\epsilon}\frac{d}{d+2}\left(\frac{b}{2R_c}\right)^2\right] + \frac{d}{d-2\epsilon}\Gamma^2(1+\epsilon)\right\} \quad (41)$$
$$=2\left\{\ln\left(\frac{2R_c}{b}\right)\left[1+\frac{d}{d+2}\left(\frac{b}{2R_c}\right)^2\right]+\frac{1}{d}+\Psi(1)\right\}.$$

The region $b\gg R_c$ is much more difficult to analyze. The asymptotics of $I(b)$ is calculated in the appendix and the result is

$$I(b)\Big|_{b\gg R_c}\simeq\left(\frac{2R_c}{b}\right)^d\frac{\Gamma(1+d/2)}{\Gamma(1-d/2)} \times\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[-\frac{\Gamma(1-d/2)}{\Gamma(1-d/2+\epsilon)}+\frac{\Gamma(d/2-\epsilon)}{\Gamma(d/2)}\right] =\left(\frac{2R_c}{b}\right)^d\Gamma\left(\frac{d}{2}\right)\Gamma\left(1+\frac{d}{2}\right)\cos\frac{\pi d}{2}. \quad (42)$$

From (29), (35) and (36) it follows that

$$\chi(b)=\chi_0(b)+\chi_{\mathrm{mass}}(b)\,, \quad (43)$$

where

$$\chi_0(b)=\frac{1}{4\pi s}\int_0^\infty q_\perp\mathrm{d}q_\perp J_0(bq_\perp)A^{\mathrm{B}}_0(s,-q_\perp^2) \quad (44)$$

and

$$\chi_{\mathrm{mass}}(b)=\left(\frac{b_c}{b}\right)^d-G_{\mathrm{N}}s\,\frac{\pi^{d/2-1}}{8\Gamma(1+d/2)}\,I(b)\,. \quad (45)$$

The asymptotics of $I(b)$ at small and large b are calculated above (see (41) and (42)). The zero mode graviton contribution to the Born amplitude is

$$A^{\mathrm{B}}_0(s,t)=\frac{G_{\mathrm{N}}s^2}{-t}\,. \quad (46)$$

Correspondingly, the eikonal amplitude is represented by the expression

$$A^{\mathrm{eik}}(s,t)=-4\pi\mathrm{i}s\int_0^\infty b\,\mathrm{d}bJ_0(b\sqrt{-t})\left[\mathrm{e}^{\mathrm{i}(\chi_0(b)+\chi_{\mathrm{mass}}(b))}-1\right]. \quad (47)$$

The massless graviton contribution to the eikonal (44) is divergent due to the Coulomb-like pole in t (46). In order to regularize it, let us assume that *the colliding particles are confined to a $(4+\delta)$-dimensional brane* with $\delta > 0$, while gravity propagates in $(4+\delta+d)$ dimensions. Then, instead of (44) we will have

$$\chi_0(b,\delta) = \frac{1}{2s}\int \frac{\mathrm{d}^{2+\delta}q_\perp}{(2\pi)^{2+\delta}}\,\mathrm{e}^{ibq_\perp} A_0^{\mathrm{B}}(s,-q_\perp^2,\delta)\,, \qquad (48)$$

where $A_0^{\mathrm{B}}(s,t,\delta) = G_{4+\delta}s^2/|t|$ and $G_{4+\delta}$ is a gravitational constant in $(4+\delta)$ dimensions. The "massive" part of the eikonal, $\chi_{\mathrm{mass}}(b,\delta)$, is analogously determined via $A_{\mathrm{mass}}(s,t,\delta)$.

We define the (eikonal) amplitude of the scattering in four dimensions as the limit

$$A^{\mathrm{eik}}(s,t) = \lim_{\delta\to 0} A^{\mathrm{eik}}(s,t,\delta)\,, \qquad (49)$$

where the eikonal amplitude,

$$A^{\mathrm{eik}}(s,t,\delta) = -2\mathrm{i}s\int \mathrm{d}^{2+\delta}b\,\mathrm{e}^{ibq_\perp}\left[\mathrm{e}^{\mathrm{i}(\chi_0(b,\delta)+\chi_{\mathrm{mass}}(b,\delta))}-1\right], \qquad (50)$$

can be rewritten by adding and subtracting the "massless" term:

$$A^{\mathrm{eik}}(s,t,\delta) = -2\mathrm{i}s\int \mathrm{d}^{2+\delta}b\,\mathrm{e}^{ibq_\perp}\,\mathrm{e}^{\mathrm{i}\chi_0(b,\delta)}\left[\mathrm{e}^{\mathrm{i}\chi_{\mathrm{mass}}(b,\delta)}-1\right] - 2\mathrm{i}s\int \mathrm{d}^{2+\delta}b\,\mathrm{e}^{ibq_\perp}\left[\mathrm{e}^{\mathrm{i}\chi_0(b,\delta)}-1\right]. \qquad (51)$$

It is easy to check that $\chi_{\mathrm{mass}}(t,\delta)$ is non-singular at $\delta = 0$. As for $\chi_0(t,\delta)$, at small δ we get from (48)

$$\chi_0(b,\delta)\Big|_{\delta\to 0} = \frac{G_{\mathrm{N}}s}{4\pi}\left[\frac{1}{\delta} - \ln(bM_{\mathrm{Pl}}) + \mathrm{O}(\delta)\right]. \qquad (52)$$

The first integral in the RHS of (51) converges at $b = 0$ ($\chi_{\mathrm{mass}}(b) \sim Ab^{-d} + B\ln(1/b)$, if $b \to 0$), and it is well defined at $b = \infty$, if $d > 2$ ($\chi_{\mathrm{mass}}(b) \sim Cb^{-d}$, if $b \to \infty$).

The second integral in the RHS of (51) is well known. Let us put

$$\frac{1}{\delta} = \ln\left(\frac{M_{\mathrm{Pl}}}{\mu_{\mathrm{IR}}}\right), \qquad (53)$$

where μ_{IR} is an infrared regulator at $\delta \to 0$. Then this integral is given by the expression for $A^{\mathrm{eik}}_{\mathrm{HVV}}(s,t)$ (16) with the replacement $\alpha_G \to \alpha_G/8\pi$ (or, equivalently, $G_{\mathrm{N}} \to G_{\mathrm{N}}/8\pi$; see our remarks after formula (19)). As a result, we obtain

$$A^{\mathrm{eik}}(s,t) = \left(\frac{4\mu_{\mathrm{IR}}^2}{-t}\right)^{-\mathrm{i}\alpha_G(s)/8\pi} \times \left\{ A_0^{\mathrm{B}}(s,t)\,\frac{\Gamma(1-\mathrm{i}\alpha_G(s)/8\pi)}{\Gamma(1+\mathrm{i}\alpha_G(s)/8\pi)} - 4\pi\mathrm{i}\frac{s}{-t}\int_0^\infty \mathrm{d}x\,x\left(\frac{x}{2}\right)^{-\mathrm{i}\alpha_G(s)/4\pi} \times J_0(x)\left[\exp\left(\mathrm{i}\chi_{\mathrm{mass}}\left(\frac{x}{\sqrt{|t|}}\right)\right)-1\right]\right\}. \qquad (54)$$

Here $\chi_{\mathrm{mass}}(b)$ is defined by formula (45) and $A_0^{\mathrm{B}}(s,t)$ is the singular part of the Born amplitude (46).

It follows directly from (34) and (30) that

$$\chi_{\mathrm{mass}}(b)\Big|_{d\to 0} \to 0\,. \qquad (55)$$

As can be seen from (26), (36) and (41), the small b behavior of $\chi_{\mathrm{mass}}(b)$ is consistent with (55). Large b asymptotics of $\chi_{\mathrm{mass}}(b)$ also obeys this limit (see (59)). Thus, by taking the limit $d \to 0$ in (54), we reproduce the well-known four-dimensional result (16) derived in [5,6].

Introducing a four-dimensional phase

$$\phi_4 = \frac{G_{\mathrm{N}}s}{8\pi}\ln\left(\frac{-t}{4\mu_{\mathrm{IR}}^2}\right), \qquad (56)$$

we get *our final result*:

$$A^{\mathrm{eik}}(s,t) = s\,\mathrm{e}^{\mathrm{i}\phi_4}\left\{\frac{G_{\mathrm{N}}s}{-t}\,\frac{\Gamma(1-\mathrm{i}G_{\mathrm{N}}s/8\pi)}{\Gamma(1+\mathrm{i}G_{\mathrm{N}}s/8\pi))} - 16\pi\mathrm{i}\,R_{\mathrm{c}}^2(R_{\mathrm{c}}\sqrt{|t|})^{-\mathrm{i}G_{\mathrm{N}}s/4\pi} \times \int_0^\infty \mathrm{d}z\,z^{1-\mathrm{i}G_{\mathrm{N}}s/4\pi}J_0(2R_{\mathrm{c}}\sqrt{|t|}\,z)\left[\mathrm{e}^{\mathrm{i}\chi_{\mathrm{mass}}(z)}-1\right]\right\}, \qquad (57)$$

with

$$\chi_{\mathrm{mass}}(z)\Big|_{z\ll 1} \simeq G_{\mathrm{N}}s\frac{\pi^{d/2-1}\Gamma(d/2)}{8}z^{-d} - G_{\mathrm{N}}s\,\frac{\pi^{d/2-1}}{4\Gamma(1+d/2)} \times\left\{\ln\left(\frac{1}{z}\right)\left[1+\frac{d}{d+2}\,z^2\right]+\frac{1}{d}+\Psi(1)\right\} \qquad (58)$$

and

$$\chi_{\mathrm{mass}}(z)\Big|_{z\gg 1} = G_{\mathrm{N}}s\frac{\pi^{d/2-1}\Gamma(d/2)}{4}z^{-d}\sin^2\left(\frac{\pi d}{4}\right). \qquad (59)$$

We have introduced the dimensionless variable $z = b/2R_{\mathrm{c}}$. The amplitude $A^{\mathrm{eik}}(s,t)$ is well defined for all $d \geqq 0$. Note that the asymptotic behavior of $\chi(b)$ (59) differs from the corresponding asymptotics of χ_{GRW}, see (24), by a factor $2\sin^2(\pi d/4)$.

Our expression (57) has an infinite phase (56). It was shown many years ago [18] that in quantum gravity each different particle pair in the initial (or final) state contributes a divergent phase factor to the S-matrix.

Note that the second term in (57) is regular at $t = 0$. Thus, for small t (namely, at $|t|R_{\mathrm{c}}^2 \ll 1$) the main contribution to the eikonal amplitude comes from the Born pole (the first term in (57)). It is interesting that this term does not depend on the compactification radius R_{c}^2 nor on the number of extra dimensions d. In other words, it is entirely four-dimensional.

The large t behavior is determined by small values of the variable z in the integral (57). Taking into account the asymptotics of $\chi_{\rm mass}(z)$ at $z \approx 0$ (58), one can obtain by using the stationary-phase technique ($d > 0$)

$$A^{\rm eik}(s,t)\Big|_{|t|R_{\rm c}^2 \gg 1} \simeq -4\pi {\rm i} s\, {\rm e}^{{\rm i}\phi_4} \frac{1}{|t|} \frac{2^{{\rm i}G_{\rm N}s/4\pi}}{\sqrt{1+d}} \times \left[\left(d \left(b_{\rm c}\sqrt{|t|} \right)^d \right)^{1/(d+1)} \right]^{1-{\rm i}G_{\rm N}s/4\pi} \times \exp\left[{\rm i}(1+d) \left(\frac{b_{\rm c}\sqrt{|t|}}{d} \right)^{d/(d+1)} \right] . \qquad (60)$$

Formula (57) has the correct physical limits. In the case when the compactification radius $R_{\rm c}$ tends to zero and, consequently, the KK graviton excitations become very heavy ($m_n \to \infty$; see (3)) and decouple from the brane particles, they make no contribution to the amplitude (since $\chi_{\rm mass} \to 0$), but a renormalized Born amplitude still is present in (57). The same is true for $d \to 0$ (no extra dimensions and, consequently, no massive gravitons are present in nature).

On the other hand, the expression for $A^{\rm eik}_{\rm GRW}(s,t)$ obtained in [12] (see formulae (22), (24) and (25)), results in $A^{\rm eik}_{\rm GRW}(s,t) = 0$ in the limit $R_{\rm c} \to 0$.

4 Discussion

In the present paper the eikonal scattering amplitude of two *brane particles*, interacting by gravity forces, is calculated in the ADD model. To go beyond the approximation of effectively infinite $R_{\rm c}$, used in other papers, we pay particular attention to the account of the contribution from the massless graviton mode. We have shown that the brane amplitude, $A^{\rm eik}$, has *both "massless" (Coulomb) and "massive" terms*. Our main result is formula (57), where $\chi_{\rm mass}(b)$ represents the contribution from the KK graviton modes with $n \geqq 1$. The expression for $\chi_{\rm mass}(b)$ and its asymptotic behavior are presented in (45), (37) and (58), and (59), respectively. Our formula (57) gives the correct four-dimensional result at both $D \to 4$ and $R_{\rm c} \to 0$.

It is interesting to compare our results with those describing a collision of two *bulk particles* in D dimensions with $D > 4$. First of all, $\chi^{\rm eik}_{\rm ACV}$ has an imaginary part, while our $\chi^{\rm eik}$ does not. The imaginary part appears in $\chi^{\rm eik}$, when one sums multiple exchange of *reggeized gravitons* [4,9].

The asymptotics of our eikonal at large impact parameter (59) coincides with the real part of $\chi^{\rm eik}_{\rm ACV}$ (11), up to a constant depending on the number of the extra dimensions.

The D-dimensional eikonal amplitude, $A^{\rm eik}_{\rm ACV}$, see (12), has the non-renormalized Born pole at $t = 0$. In the brane amplitude (57) the renormalized Born pole is reproduced. This singular part makes a leading contribution at small momentum transfers, and it coincides with the D-dimensional amplitude taken at $D \to 4$.

The presence of the compact extra dimensions does not influence the small t behavior of the scattering amplitude, if a collision takes place on the (1+3)-dimensional brane. This is easy to understand, since, from the point of view of four dimensions, the higher space dimensions supply us with the KK tower of *massive exchange quanta* (in our case, massive gravitons). These new massive quanta cannot hide the long-range forces originating from the massless graviton.

Thus, accounting for the contributions of the massive gravitons results in an additive term, which is important at large and intermediate t (60). Note that the eikonal depends, in general, on the ratio $b/2R_{\rm c}$ (it is of the form $\chi(b) \sim (b_{\rm c}/b)^d$ only at $b \to 0, \infty$). Therefore, *the relevant dimensionless parameters for the amplitude are $G_{\rm N}s$ and $R_{\rm c}^2|t|$* (but not $b_{\rm c}^2|t|$). Recall that in flat $(4+d)$ dimensions the eikonal amplitude is given in terms of the dimensional parameter $G_D s|t|^{d/2}$ (see the formulae in Sect. 2).

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Appendix

In this appendix we will calculate the asymptotics of the RHS of (40) at large values of $b/2R_{\rm c}$. The quantity under consideration, $I(b)$, is represented in terms of the generalized hypergeometric functions, ${}_1F_2\,(\alpha_1;\, \beta_1, \beta_2;\, z)$ and ${}_2F_3\,(\alpha_1, \alpha_2;\, \beta_1, \beta_2, \beta_3;\, z)$; see (38) and (40). Let us introduce the notation

$${}_pF_q \left(\begin{matrix} \alpha_p \\ \beta_q \end{matrix} \middle| \, z \right) \equiv {}_pF_q\,(\alpha_1, \dots \alpha_p;\, \beta_1, \dots \beta_q;\, z)\ ; \qquad (A.1)$$

$$\Gamma(\alpha_p) \equiv \Gamma(\alpha_1)\Gamma(\alpha_2)\dots\Gamma(\alpha_p)\,. \qquad (A.2)$$

Sometimes we will write simply ${}_pF_q(z)$.

To solve the problem, we need to use the full asymptotic expansion of ${}_pF_{p+1}(z)$ at large z [19]:

$${}_pF_{p+1} \left(\begin{matrix} \alpha_p \\ \beta_{p+1} \end{matrix} \middle| \, \frac{z^2}{4} \right) \simeq \frac{\Gamma(\beta_{p+1})}{\Gamma(\alpha_p)} \left\{ \left\{ K_{p,p+1}\left[\left(\frac{1}{2} z \right)^2 \right] + K_{p,p+1}\left[\left(\frac{1}{2} z {\rm e}^{{\rm i}\pi} \right)^2 \right] + L_{p,p+1}\left[\left(\frac{1}{2} z {\rm e}^{{\rm i}\pi} \right)^2 \right] \right\} ; \qquad (A.3)$$

$|z| \to \infty$, $\arg z = 0$. Let us recall that $z = b/R_{\rm c} > 0$ and $p = 1, 2$ in our case. The function $K_{p,p+1}(z)$ in (A.3) is given by the series in inverse powers of the variable z [19]:

$$K_{p,p+1}\left[\left(\frac{1}{2} z \right)^2 \right] = \frac{1}{2^{2\gamma+1}\sqrt{\pi}} {\rm e}^z z^{2\gamma} \sum_{k=0}^{\infty} d_k z^{-k}\,, \quad d_0 = 1\,, \qquad (A.4)$$

with

$$\gamma = \frac{1}{2} \left(\frac{1}{2} + \sum_{n=1}^{p} \alpha_n - \sum_{n=1}^{p+1} \beta_n \right) . \qquad (A.5)$$

Thus, $K_{p,p+1}(z)$ increases exponentially in z at $z \to +\infty$, which can result in an exponential rise of I_ϵ (40) in the impact parameter b. However, we will show now that this is not the case due to *a complete cancellation* of terms, proportional to $K_{1,2}(z)$ and $z^{\epsilon}K_{2,3}(z)$ in (40).

Let us note first that $\gamma = -3/4$ and $\gamma = -3/4 - \epsilon$, respectively, for ${}_1F_2\,(d/2-\epsilon; 1+d/2-\epsilon, 1;\, z)$ and ${}_2F_3\,(d/2, 1; 1+d/2, 1+\epsilon, 1+\epsilon;\, z)$. For $p = 1$, the coefficients d_k in (A.4) obey the recursion formulae [19]

$$\begin{aligned} 2(k+1)\, d_{k+1}^{(p=1)} &= \left[3k^2 + 2k(1 + C_1 - 3B_1) + 4D_1\right] d_k^{(p=1)} \\ &\quad -(k-2\gamma-1)(k-2\gamma+1-2\beta_1) \\ &\qquad \times(k-2\gamma+1-2\beta_2) d_{k-1}^{(p=1)} \\ &= \tilde{A}_k^{(p=1)} d_k^{(p=1)} + \tilde{A}_{k-1}^{(p=1)} d_{k-1}^{(p=1)} \,, \end{aligned} \tag{A.6}$$

where we have introduced the notation

$$B_1 = \sum_{n=1}^{p} \alpha_n \,, \qquad C_1 = \sum_{n=1}^{p+1} \beta_n \,, \tag{A.7}$$

$$B_2 = \sum_{n=2}^{p} \sum_{m=1}^{n-1} \alpha_n \alpha_m \,, \qquad C_2 = \sum_{n=2}^{p+1} \sum_{m=1}^{n-1} \beta_n \beta_m \,, \tag{A.8}$$

$$D_1 = C_2 - B_2 + \frac{1}{4}(B_1 - C_1)(3B_1 + C_1 - 2) - \frac{3}{16} \,. \tag{A.9}$$

For $p = 2$, the recursion relations look like [19]

$$\begin{aligned} 2(k+1)\, d_{k+1}^{(p=2)} &= \left[5k^2 + 2k(3 + B_1 - 3C_1 - 10\gamma) + 4D_1\right] d_k^{(p=2)} \\ &\quad -[4k^3 - 6k^2(C_1 + 4\gamma) \\ &\quad +2k(24\gamma^2 + 12\gamma C_1 + C_1 + 4C_2 - 1) \\ &\quad -32\gamma^3 - 24\gamma^2 C_1 - 4\gamma(C_1 + 4C_2 - 1) \\ &\qquad +2C_1 - 4C_2 - 8C_3 - 1] d_{k-1}^{(p=2)} \\ &\quad +(k-2\gamma-2)(k-2\gamma-2\beta_1) \\ &\qquad \times(k-2\gamma-2\beta_2)(k-2\gamma-2\beta_3)\, d_{k-2}^{(p=2)} \\ &= A_k^{(p=2)} d_k^{(p=2)} + A_{k-1}^{(p=2)} d_{k-1}^{(p=2)} + A_{k-2}^{(p=2)} d_{k-2}^{(p=2)} \,, \end{aligned} \tag{A.10}$$

where

$$C_3 = \beta_1 \beta_2 \beta_3 \,, \tag{A.11}$$

and the other quantities are defined as before; see (A.7)–(A.9).

By making the replacement $k \to k-1$ in (A.6), we get

$$\begin{aligned} 2k\, d_k^{(p=1)} &= \left[3(k-1)^2 + 2(k-1)(1 + C_1 - 3B_1) + 4D_1\right] d_{k-1}^{(p=1)} \\ &\quad -(k-2\gamma-2)(k-2\gamma-2\beta_1)(k-2\gamma-2\beta_2) d_{k-2}^{(p=1)} \,. \end{aligned} \tag{A.12}$$

Let us now rewrite (A.6) in the form

$$\begin{aligned} 2(k+1)\, d_{k+1}^{(p=1)} &= \left[\tilde{A}_k^{(p=1)} - A_k^{(p=2)}\right] d_k^{(p=1)} \\ &\quad + A_k^{(p=2)} d_k^{(p=1)} + \tilde{A}_{k-1}^{(p=1)} d_{k-1}^{(p=1)} \,, \end{aligned} \tag{A.13}$$

and substitute $d_k^{(p=1)}$ from (A.12) into the first term in the RHS of (A.13). Then we obtain

$$2(k+1) d_{k+1}^{(p=1)} = A_k^{(p=1)} d_k^{(p=1)} + A_{k-1}^{(p=1)} d_{k-1}^{(p=1)} + A_{k-2}^{(p=1)} d_{k-2}^{(p=1)} \,. \tag{A.14}$$

By direct calculations one can check that $A_i^{(p=1)} = A_i^{(p=2)}$, namely

$$\begin{aligned} A_k^{(p=1)} &= A_k^{(p=2)} \\ &= 5k^2 + k(5 - 2n + 8\epsilon) + \frac{13}{4} - 2n + 4\epsilon \,, \\ A_{k-1}^{(p=1)} &= A_{k-1}^{(p=2)} \\ &= -4k^3 + 3k^2(n - 4\epsilon) + k(-1 + 4n\epsilon - 8\epsilon^2) \\ &\quad + \frac{1}{4}(n - 4\epsilon) \,, \\ A_{k-2}^{(p=1)} &= A_{k-2}^{(p=2)} \\ &= \frac{1}{16}(2k-1)^2(2k - 1 + 4\epsilon) \\ &\qquad \times(2k - 1 - 2n + 4\epsilon) \,. \end{aligned} \tag{A.15}$$

In other words, we have shown that $d_k^{(p=1)}$ and $d_k^{(p=2)}$ obey the same recursion formulae, and $d_k^{(p=1)} = d_k^{(p=2)}$ *for all* k in the series (A.4).

Therefore, the K-functions do not contribute to the expansion (A.3) in our case, and we have

$$\begin{aligned} I_\epsilon(b) &= \frac{d}{2\epsilon}\, \Gamma(1+\epsilon)\Gamma(1+\epsilon) \\ &\quad \times \left[-\left(\frac{b^2}{4R_c^2}\right)^{\epsilon} L_{2,3}\left(\begin{matrix} d/2,\ 1 \\ 1+d/2,\ 1+\epsilon,\ 1+\epsilon \end{matrix} \middle| \frac{b^2}{4R_c^2} \right) \right. \\ &\qquad \left. + L_{1,2}\left(\begin{matrix} d/2-\epsilon \\ 1+d/2-\epsilon, ,1 \end{matrix} \middle| \frac{b^2}{4R_c^2} \right) \right] . \end{aligned} \tag{A.16}$$

Note that the L-function is related to the Meijer G-function [19]:

$$\begin{aligned} L_{p,p+1}\left(\begin{matrix} \alpha_1, \dots \alpha_p \\ \beta_1, \dots \beta_{p+1} \end{matrix} \middle| z \right) = G_{p+2,\,p}^{p,\,1}\left(\frac{1}{z} \middle| \begin{matrix} 1,\ \beta_1, \dots \beta_{p+1} \\ \alpha_1, \dots \alpha_p \end{matrix} \right) . \end{aligned} \tag{A.17}$$

In its turn, the G-function can be represented as a series in generalized hypergeometric functions of an inverse power of z [16]:

$$L_{p,p+1}(z) = \sum_{n=1}^{p} L_{p,p+1}^{(n)}(z) \,, \tag{A.18}$$

where

$$\begin{aligned} L_{p,p+1}^{(n)}(z) &= z^{-\alpha_n} \frac{\Gamma(\alpha_n)\Gamma(\alpha_p - \alpha_n)^*}{\Gamma(\beta_{p+1} - \alpha_n)} \\ &\quad \times {}_{p+2}F_{p-1}\left(\begin{matrix} \alpha_n,\ 1+\alpha_n-\beta_{p+1} \\ 1+\alpha_n-\alpha_p^* \end{matrix} \middle| -\frac{1}{z} \right) \end{aligned} \tag{A.19}$$

(the superscript * means that the term with $\alpha_n = \alpha_p$ is not included in the product of Γ-functions).

As a result, we arrive at the expression for I_ϵ, which is convenient for analyzing its large b behavior:

$$I_\epsilon(b) = \frac{1}{\epsilon}\Gamma(1-\epsilon)\Gamma(1+\epsilon)\Gamma(1+d/2) \times \left\{ -\left(\frac{b^2}{4R_c^2}\right)^{\epsilon-1} \frac{\Gamma(d/2-1)}{\Gamma^2(\epsilon)\Gamma^2(d/2)} \times {}_4F_1\left(1, 1-\frac{d}{2}, 1-\epsilon, 1-\epsilon; 2-\frac{d}{2}; -\frac{4R_c^2}{b^2}\right) + \left(\frac{b^2}{4R_c^2}\right)^{\epsilon-d/2} \frac{1}{\Gamma(1-d/2+\epsilon)} \times \left[-\frac{\Gamma(1-d/2)}{\Gamma(1-d/2+\epsilon)} + \frac{\Gamma(d/2-\epsilon)}{\Gamma(d/2)}\right] \right\}. \tag{A.20}$$

Up to now, we did not consider the parameter ϵ to be small. Finally, the desired asymptotics looks like

$$I(b)\Big|_{b\gg R_c} = \lim_{\epsilon\to 0} I_\epsilon\Big|_{b\gg R_c} = \left(\frac{2R_c}{b}\right)^d \frac{\Gamma(1+d/2)}{\Gamma(1-d/2)} \times \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[-\frac{\Gamma(1-d/2)}{\Gamma(1-d/2+\epsilon)} + \frac{\Gamma(d/2-\epsilon)}{\Gamma(d/2)}\right]. \tag{A.21}$$

By expanding the RHS of equality (A.21) in ϵ and taking the limit $\epsilon \to 0$, we derive the asymptotic formula for $I(b)$ presented in the text; see (42).

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